ISSN 1464-8997 (on line) 1464-8989 (printed)

Geometry & Topology Monographs
Volume 3: Invitation to higher local fields
Part I, section 17, pages 143–150

17. An approach to higher ramification theory

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We use the notation of sections 1 and 10.

17.0. Approach of Hyodo and Fesenko

Let K be an n-dimensional local field, L/K a finite abelian extension. Define a filtration on $\mathrm{Gal}(L/K)$ (cf. [H], [F, sect. 4]) by

$$\mathrm{Gal}(L/K)^{\mathbf{i}} = \Upsilon^{-1}_{L/K}(U_{\mathbf{i}}K^{\mathrm{top}}_n(K) + N_{L/K}K^{\mathrm{top}}_n(L)/N_{L/K}K^{\mathrm{top}}_n(L)), \quad \mathbf{i} \in \mathbb{Z}^n_+,$$

where $U_{\mathbf{i}}K^{\mathrm{top}}_n(K) = \{U_{\mathbf{i}}\} \cdot K^{\mathrm{top}}_{n-1}(K)$, $U_{\mathbf{i}} = 1 + P_K(\mathbf{i})$,

$$\Upsilon^{-1}_{L/K}\colon K^{\mathrm{top}}_n(K)/N_{L/K}K^{\mathrm{top}}_n(L) \xrightarrow{\sim} \mathrm{Gal}(L/K)$$

is the reciprocity map.

Then for a subextension M/K of L/K

$$\mathrm{Gal}(M/K)^{\mathbf{i}} = \mathrm{Gal}(L/K)^{\mathbf{i}}\,\mathrm{Gal}(L/M)/\,\mathrm{Gal}(L/M)$$

which is a higher dimensional analogue of Herbrand's theorem. However, if one defines a generalization of the Hasse–Herbrand function and lower ramification filtration, then for $n > 1$ the lower filtration on a subgroup does not coincide with the induced filtration in general.

Below we shall give another construction of the ramification filtration of L/K in the two-dimensional case; details can be found in [Z], see also [KZ]. This construction can be considered as a development of an approach by K. Kato and T. Saito in [KS].

Definition. Let K be a complete discrete valuation field with residue field k_K of characteristic p. A finite extension L/K is called *ferociously ramified* if $|L : K| = |k_L : k_K|_{\mathrm{ins}}$.

In addition to the nice ramification theory for totally ramified extensions, there is a nice ramification theory for ferociously ramified extensions L/K such that k_L/k_K is generated by one element; the reason is that in both cases the ring extension $\mathcal{O}_L/\mathcal{O}_K$ is monogenic, i.e., generated by one element, see section 18.

17.1. Almost constant extensions

Everywhere below K is a complete discrete valuation field with residue field k_K of characteristic p such that $|k_K : k_K^p| = p$. For instance, K can be a two-dimensional local field, or $K = \mathbb{F}_q(X_1)((X_2))$ or the quotient field of the completion of $\mathbb{Z}_p[T]_{(p)}$ with respect to the p-adic topology.

Definition. For the field K define a base (sub)field B as

$B = \mathbb{Q}_p \subset K$ if $\operatorname{char}(K) = 0$,

$B = \mathbb{F}_p((\rho)) \subset K$ if $\operatorname{char}(K) = p$, where ρ is an element of K with $v_K(\rho) > 0$.

Denote by k_0 the completion of $B(\mathcal{R}_K)$ inside K. Put $k = k_0^{\mathrm{alg}} \cap K$.

The subfield k is a maximal complete subfield of K with perfect residue field. It is called a *constant subfield* of K. A constant subfield is defined canonically if $\operatorname{char}(K) = 0$. Until the end of section 17 we assume that B (and, therefore, k) is fixed.

By v we denote the valuation $K^{\mathrm{alg}*} \to \mathbb{Q}$ normalized so that $v(B^*) = \mathbb{Z}$.

Example. If $K = F\{\{T\}\}$ where F is a mixed characteristic complete discrete valuation field with perfect residue field, then $k = F$.

Definition. An extension L/K is said to be *constant* if there is an algebraic extension l/k such that $L = Kl$.

An extension L/K is said to be *almost constant* if $L \subset L_1L_2$ for a constant extension L_1/K and an unramified extension L_2/K.

A field K is said to be *standard*, if $e(K|k) = 1$, and *almost standard*, if some finite unramified extension of K is a standard field.

Epp's theorem on elimination of wild ramification. ([E], [KZ]) *Let L be a finite extension of K. Then there is a finite extension k' of a constant subfield k of K such that $e(Lk'|Kk') = 1$.*

Corollary. *There exists a finite constant extension of K which is a standard field.*

Proof. See the proof of the Classification Theorem in 1.1.

Lemma. *The class of constant (almost constant) extensions is closed with respect to taking compositums and subextensions. If L/K and M/L are almost constant then M/K is almost constant as well.*

Definition. Denote by L_c the maximal almost constant subextension of K in L.

Properties.
(1) Every tamely ramified extension is almost constant. In other words, the (first) ramification subfield in L/K is a subfield of L_c.
(2) If L/K is normal then L_c/K is normal.
(3) There is an unramified extension L_0' of L_0 such that $L_c L_0'/L_0$ is a constant extension.
(4) There is a constant extension L_c'/L_c such that LL_c'/L_c' is ferociously ramified and $L_c' \cap L = L_c$. This follows immediately from Epp's theorem.

The principal idea of the proposed approach to ramification theory is to split L/K into a tower of three extensions: L_0/K, L_c/L_0, L/L_c, where L_0 is the inertia subfield in L/K. The ramification filtration for $\mathrm{Gal}(L_c/L_0)$ reflects that for the corresponding extensions of constants subfields. Next, to construct the ramification filtration for $\mathrm{Gal}(L/L_c)$, one reduces to the case of ferociously ramified extensions by means of Epp's theorem. (In the case of higher local fields one can also construct a filtration on $\mathrm{Gal}(L_0/K)$ by lifting that for the first residue fields.)

Now we give precise definitions.

17.2. Lower and upper ramification filtrations

Keep the assumption of the previous subsection. Put

$$\mathcal{A} = \{-1, 0\} \cup \{(\mathfrak{c}, s) : 0 < s \in \mathbb{Z}\} \cup \{(\mathfrak{i}, r) : 0 < r \in \mathbb{Q}\}.$$

This set is linearly ordered as follows:

$$-1 < 0 < (\mathfrak{c}, i) < (\mathfrak{i}, j) \text{ for any } i, j;$$
$$(\mathfrak{c}, i) < (\mathfrak{c}, j) \text{ for any } i < j;$$
$$(\mathfrak{i}, i) < (\mathfrak{i}, j) \text{ for any } i < j.$$

Definition. Let $G = \mathrm{Gal}(L/K)$. For any $\alpha \in \mathcal{A}$ we define a subgroup G_α in G.

Put $G_{-1} = G$, and denote by G_0 the inertia subgroup in G, i.e.,

$$G_0 = \{g \in G : v(g(a) - a) > 0 \text{ for all } a \in \mathcal{O}_L\}.$$

Let L_c/K be constant, and let it contain no unramified subextensions. Then define

$$G_{\mathfrak{c},i} = \mathrm{pr}^{-1}(\mathrm{Gal}(l/k)_i)$$

where l and k are the constant subfields in L and K respectively,

$$\mathrm{pr}\colon\ \mathrm{Gal}(L/K) \to \mathrm{Gal}(l/k) = \mathrm{Gal}(l/k)_0$$

is the natural projection and $\mathrm{Gal}(l/k)_i$ are the classical ramification subgroups. In the general case take an unramified extension K'/K such that $K'L/K'$ is constant and contains no unramified subextensions, and put $G_{\mathfrak{c},i} = \mathrm{Gal}(K'L/K')_{\mathfrak{c},i}$.

Finally, define $G_{\mathfrak{i},i}$, $i > 0$. Assume that L_c is standard and L/L_c is ferociously ramified. Let $t \in \mathcal{O}_L$, $\bar{t} \notin k_L^p$. Define

$$G_{\mathfrak{i},i} = \{g \in G : v(g(t) - t) \geqslant i\}$$

for all $i > 0$.

In the general case choose a finite extension l'/l such that $l'L_c$ is standard and $e(l'L|l'L_c) = 1$. Then it is clear that $\mathrm{Gal}(l'L/l'L_c) = \mathrm{Gal}(L/L_c)$, and $l'L/l'L_c$ is ferociously ramified. Define

$$G_{\mathfrak{i},i} = \mathrm{Gal}(l'L/l'L_c)_{\mathfrak{i},i}$$

for all $i > 0$.

Proposition. *For a finite Galois extension L/K the lower filtration $\{\mathrm{Gal}(L/K)_\alpha\}_{\alpha \in \mathcal{A}}$ is well defined.*

Definition. Define a generalization $h_{L/K}\colon \mathcal{A} \to \mathcal{A}$ of the Hasse–Herbrand function. First, we define

$$\Phi_{L/K} \colon \mathcal{A} \to \mathcal{A}$$

as follows:

$$\begin{aligned} \Phi_{L/K}(\alpha) &= \alpha \quad \text{for } \alpha = -1, 0; \\ \Phi_{L/K}((\mathfrak{c}, i)) &= \left(\mathfrak{c}, \frac{1}{e(L|K)} \int_0^i |\,\mathrm{Gal}(L_c/K)_{\mathfrak{c},t}| dt\right) \quad \text{for all } i > 0; \\ \Phi_{L/K}((\mathfrak{i}, i)) &= \left(\mathfrak{i}, \int_0^i |\,\mathrm{Gal}(L/K)_{\mathfrak{i},t}| dt\right) \quad \text{for all } i > 0. \end{aligned}$$

It is easy to see that $\Phi_{L/K}$ is bijective and increasing, and we introduce

$$h_{L/K} = \Psi_{L/K} = \Phi_{L/K}^{-1}.$$

Define the upper filtration $\mathrm{Gal}(L/K)^\alpha = \mathrm{Gal}(L/K)_{h_{L/K}(\alpha)}$.

All standard formulas for intermediate extensions take place; in particular, for a normal subgroup H in G we have $H_\alpha = H \cap G_\alpha$ and $(G/H)^\alpha = G^\alpha H/H$. The latter relation enables one to introduce the upper filtration for an infinite Galois extension as well.

Remark. The filtrations do depend on the choice of a constant subfield (in characteristic p).

Example. Let $K = \mathbb{F}_p((t))((\pi))$. Choose $k = B = \mathbb{F}_p((\pi))$ as a constant subfield. Let $L = K(b)$, $b^p - b = a \in K$. Then

if $a = \pi^{-i}$, i prime to p, then the ramification break of $\mathrm{Gal}(L/K)$ is $(\mathfrak{c}, i)$;
if $a = \pi^{-pi}t$, i prime to p, then the ramification break of $\mathrm{Gal}(L/K)$ is $(\mathfrak{i}, i)$;
if $a = \pi^{-i}t$, i prime to p, then the ramification break of $\mathrm{Gal}(L/K)$ is $(\mathfrak{i}, i/p)$;
if $a = \pi^{-i}t^p$, i prime to p, then the ramification break of $\mathrm{Gal}(L/K)$ is $(\mathfrak{i}, i/p^2)$.

Remark. A dual filtration on $K/\wp(K)$ is computed in the final version of [Z], see also [KZ].

17.3. Refinement for a two-dimensional local field

Let K be a two-dimensional local field with $\mathrm{char}\,(k_K) = p$, and let k be the constant subfield of K. Denote by

$$\mathbf{v} = (v_1, v_2)\colon (K^{\mathrm{alg}})^* \to \mathbb{Q} \times \mathbb{Q}$$

the extension of the rank 2 valuation of K, which is normalized so that:

- $v_2(a) = v(a)$ for all $a \in K^*$,
- $v_1(u) = w(\overline{u})$ for all $u \in U_{K^{\mathrm{alg}}}$, where w is a non-normalized extension of v_{k_K} on k_K^{alg}, and $\overline{u}$ is the residue of u,
- $\mathbf{v}(c) = (0, e(k|B)^{-1} v_k(c))$ for all $c \in k$.

It can be easily shown that $\mathbf{v}$ is uniquely determined by these conditions, and the value group of $\mathbf{v}|_{K^*}$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

Next, we introduce the index set

$$\mathcal{A}_2 = \mathcal{A} \cup \mathbb{Q}^2_+ = \mathcal{A} \cup \{(i_1, i_2) : i_1, i_2 \in \mathbb{Q}, i_2 > 0\}$$

and extend the ordering of $\mathcal{A}$ onto $\mathcal{A}_2$ assuming

$$(\mathfrak{i}, i_2) < (i_1, i_2) < (i_1', i_2) < (\mathfrak{i}, i_2')$$

for all $i_2 < i_2'$, $i_1 < i_1'$.

Now we can define G_{i_1,i_2}, where G is the Galois group of a given finite Galois extension L/K. Assume first that L_c is standard and L/L_c is ferociously ramified. Let $t \in \mathcal{O}_L$, $\overline{t} \notin k_L^p$ (e.g., a first local parameter of L). We define

$$G_{i_1,i_2} = \left\{ g \in G : \mathbf{v}\big(t^{-1}g(t) - 1\big) \geqslant (i_1, i_2) \right\}$$

for $i_1, i_2 \in \mathbb{Q}$, $i_2 > 0$. In the general case we choose l'/l (l is the constant subfield of both L and L_c) such that $l'L_c$ is standard and $l'L/l'L_c$ is ferociously ramified and put

$$G_{i_1,i_2} = \mathrm{Gal}(l'L/l'L_c)_{i_1,i_2}.$$

We obtain a well defined lower filtration $(G_\alpha)_{\alpha \in \mathcal{A}_2}$ on $G = \mathrm{Gal}(L/K)$.

In a similar way to 17.2, one constructs the Hasse–Herbrand functions $\Phi_{2,L/K}\colon \mathcal{A}_2 \to \mathcal{A}_2$ and $\Psi_{2,L/K} = \Phi_{2,L/K}^{-1}$ which extend Φ and Ψ respectively. Namely,

$$\Phi_{2,L/K}((i_1,i_2)) = \int_{(0,0)}^{(i_1,i_2)} |\operatorname{Gal}(L/K)_t|\,dt.$$

These functions have usual properties of the Hasse–Herbrand functions φ and $h = \psi$, and one can introduce an $\mathcal{A}_2$-indexed upper filtration on any finite or infinite Galois group G.

17.4. Filtration on $K^{\text{top}}(K)$

In the case of a two-dimensional local field K the upper ramification filtration for K^{ab}/K determines a compatible filtration on $K_2^{\text{top}}(K)$. In the case where $\operatorname{char}(K) = p$ this filtration has an explicit description given below.

From now on, let K be a two-dimensional local field of prime characteristic p over a quasi-finite field, and k the constant subfield of K. Introduce $\mathbf{v}$ as in 17.3. Let π_k be a prime of k.

For all $\alpha \in \mathbb{Q}_+^2$ introduce subgroups

$$Q_\alpha = \{\,\{\pi_k, u\} \,:\, u \in K, \mathbf{v}(u-1) \geqslant \alpha\,\} \subset VK_2^{\text{top}}(K);$$
$$Q_\alpha^{(n)} = \{a \in K_2^{\text{top}}(K) \,:\, p^n a \in Q_\alpha\};$$
$$S_\alpha = \operatorname{Cl} \bigcup_{n\geqslant 0} Q_{p^n\alpha}^{(n)}.$$

For a subgroup A, $\operatorname{Cl} A$ denotes the intersection of all open subgroups containing A.

The subgroups S_α constitute the heart of the ramification filtration on $K_2^{\text{top}}(K)$. Their most important property is that they have nice behaviour in unramified, constant and ferociously ramified extensions.

Proposition 1. *Suppose that K satisfies the following property.*

(*) *The extension of constant subfields in any finite unramified extension of K is also unramified.*

Let L/K be either an unramified or a constant totally ramified extension, $\alpha \in \mathbb{Q}_+^2$. Then we have $N_{L/K} S_{\alpha,L} = S_{\alpha,K}$.

Proposition 2. *Let K be standard, L/K a cyclic ferociously ramified extension of degree p with the ramification jump h in lower numbering, $\alpha \in \mathbb{Q}_+^2$. Then:*

(1) $N_{L/K} S_{\alpha,L} = S_{\alpha+(p-1)h,K}$, *if* $\alpha > h$;

(2) $N_{L/K} S_{\alpha,L}$ *is a subgroup in* $S_{p\alpha,K}$ *of index* p, *if* $\alpha \leqslant h$.

Now we have ingredients to define a decreasing filtration $\{\mathrm{fil}_\alpha K_2^{\mathrm{top}}(K)\}_{\alpha\in\mathcal{A}_2}$ on $K_2^{\mathrm{top}}(K)$. Assume first that $\widetilde{K}$ satisfies the condition (*). It follows from [KZ, Th. 3.4.3] that for some purely inseparable constant extension K'/K the field K' is almost standard. Since K' satisfies (*) and is almost standard, it is in fact standard.

Denote

$$\begin{aligned}
\mathrm{fil}_{\alpha_1,\alpha_2} K_2^{\mathrm{top}}(K) &= S_{\alpha_1,\alpha_2};\\
\mathrm{fil}_{\mathfrak{i},\alpha_2} K_2^{\mathrm{top}}(K) &= \mathrm{Cl} \bigcup_{\alpha_1\in\mathbb{Q}} \mathrm{fil}_{\alpha_1,\alpha_2} K_2^{\mathrm{top}}(K) \text{ for } \alpha_2\in\mathbb{Q}_+;\\
T_K &= \mathrm{Cl} \bigcup_{\alpha\in\mathbb{Q}_+^2} \mathrm{fil}_\alpha K_2^{\mathrm{top}}(K);\\
\mathrm{fil}_{\mathfrak{c},i} K_2^{\mathrm{top}}(K) &= T_K + \{\,\{t,u\} : u\in k,\ v_k(u-1)\geqslant i\} \text{ for all } i\in\mathbb{Q}_+,\\
&\quad \text{if } K=k\{\{t\}\} \text{ is standard};\\
\mathrm{fil}_{\mathfrak{c},i} K_2^{\mathrm{top}}(K) &= N_{K'/K}\,\mathrm{fil}_{\mathfrak{c},i} K_2^{\mathrm{top}}(K'), \text{ where } K'/K \text{ is as above};\\
\mathrm{fil}_0 K_2^{\mathrm{top}}(K) &= U(1)K_2^{\mathrm{top}}(K) + \{t,\mathcal{R}_K\}, \text{ where } U(1)K_2^{\mathrm{top}}(K) = \{1+P_K(1),K^*\},\\
&\quad t \text{ is the first local parameter};\\
\mathrm{fil}_{-1} K_2^{\mathrm{top}}(K) &= K_2^{\mathrm{top}}(K).
\end{aligned}$$

It is easy to see that for some unramified extension $\widetilde{K}/K$ the field $\widetilde{K}$ satisfies the condition (*), and we define $\mathrm{fil}_\alpha K_2^{\mathrm{top}}(K)$ as $N_{\widetilde{K}/K}\,\mathrm{fil}_\alpha K_2^{\mathrm{top}}(\widetilde{K})$ for all $\alpha\geqslant 0$, and $\mathrm{fil}_{-1} K_2^{\mathrm{top}}(K)$ as $K_2^{\mathrm{top}}(K)$. It can be shown that the filtration $\{\mathrm{fil}_\alpha K_2^{\mathrm{top}}(K)\}_{\alpha\in\mathcal{A}_2}$ is well defined.

Theorem 1. *Let L/K be a finite abelian extension, $\alpha\in\mathcal{A}_2$. Then $N_{L/K}\,\mathrm{fil}_\alpha K_2^{\mathrm{top}}(L)$ is a subgroup in $\mathrm{fil}_{\Phi_{2,L/K}(\alpha)} K_2^{\mathrm{top}}(K)$ of index $|\mathrm{Gal}(L/K)_\alpha|$. Furthermore,*

$$\mathrm{fil}_{\Phi_{L/K}(\alpha)} K_2^{\mathrm{top}}(K)\cap N_{L/K}K_2^{\mathrm{top}}(L) = N_{L/K}\,\mathrm{fil}_\alpha K_2^{\mathrm{top}}(L).$$

Theorem 2. *Let L/K be a finite abelian extension, and let*

$$\Upsilon_{L/K}^{-1}\colon K_2^{\mathrm{top}}(K)/N_{L/K}K_2^{\mathrm{top}}(L)\to \mathrm{Gal}(L/K)$$

be the reciprocity map. Then

$$\Upsilon_{L/K}^{-1}(\mathrm{fil}_\alpha K_2^{\mathrm{top}}(K) \mod N_{L/K}K_2^{\mathrm{top}}(L)) = \mathrm{Gal}(L/K)^\alpha$$

for any $\alpha\in\mathcal{A}_2$.

Remarks. 1. The ramification filtration, constructed in 17.2, does not give information about the classical ramification invariants in general. Therefore, this construction can be considered only as a provisional one.

2. The filtration on $K_2^{\mathrm{top}}(K)$ constructed in 17.4 behaves with respect to the norm map much better than the usual filtration $\{U_{\mathbf{i}} K_2^{\mathrm{top}}(K)\}_{\mathbf{i}\in\mathbb{Z}_+^n}$. We hope that this filtration can be useful in the study of the structure of K^{top}-groups.

3. In the mixed characteristic case the description of "ramification" filtration on $K_2^{\mathrm{top}}(K)$ is not very nice. However, it would be interesting to try to modify the ramification filtration on $\mathrm{Gal}(L/K)$ in order to get the filtration on $K_2^{\mathrm{top}}(K)$ similar to that described in 17.4.

4. It would be interesting to compute ramification of the extensions constructed in sections 13 and 14.

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